

Factorization of colored knot polynomials at roots of unity

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ABSTRACT

From analysis of a big variety of different knots we conclude that at q , which is an $2m$ -th root of unity, $q^{2m} = 1$, HOMFLY polynomials in symmetric representations $[r]$ satisfy recursion identity: $H_{r+m} = H_r \cdot H_m$ for any A , which is a generalization of the property $H_r = H_1^r$ for special polynomials at $m = 1$. We conjecture a further generalization to arbitrary representation R , which, however, is checked only for torus knots. Next, Kashaev polynomial, which arises from H_R at $q^2 = e^{2\pi i/|R|}$, turns equal to the special polynomial with A substituted by $A^{|R|}$, provided R is a single-hook representations (including arbitrary symmetric) – what provides a $q - A$ dual to the similar property of Alexander polynomial. All this implies non-trivial relations for the coefficients of the differential expansions, which are believed to provide reasonable coordinates in the space of knots – existence of such universal relations means that these variables are still not unconstrained.

Knot polynomials [1, 2] are Wilson loop averages in Chern-Simons theory [3] and their study provides important knowledge and intuition for understanding the properties of gauge-invariant observables in generic Yang-Mills theory. Since Chern-Simons theory is topological, the space-time dependence is completely decoupled and one can extract pure information about the representation (color) dependence. However, the problem of calculating colored HOMFLY polynomials

$$H_R^K(A, q^2) = \left\langle \text{Tr}_R \text{P exp} \left(\oint_K \mathcal{A} \right) \right\rangle^{CS} \quad (1)$$

with the gauge group $Sl(N)$ and coupling constant g converted into $q^2 = \exp \left(\frac{2\pi i}{g+N} \right)$ and $A = q^N$, turned to be highly non-trivial. Only recently considerable advances were achieved in [4, 5], based on decades of the previous work [6]-[36], opening a possibility to look for properties, that are valid universally, i.e. for arbitrary knots. In [37] we showed, how this new information leads to immediate breakthrough in the theory of differential expansions [22, 28, 31]. These expansions provide a non-trivial knot-dependent "quantization" of the archetypical factorization property [21]-[24]

$$\sigma_R^K(A) = \left(\sigma_{[1]}^K(A) \right)^{|R|} \quad (2)$$

of the *special* polynomials $\sigma_R(A) = H_R(q^2 = 1, A)$, which are restriction of HOMFLY to $q^2 = 1$, i.e. a kind of their large- N limit. Differential expansion substitutes *factorization* at $q = 1$ by *expansion* at $q \neq 1$, which, however, contains finitely many terms with their own pronounced factorization properties. They are best studied for symmetric representation $R = [r]$:

$$\mathcal{H}_r^K(A, q^2|h) = 1 + \sum_{s=1}^r \frac{[r]!}{[s]![r-s]!} \cdot h^s G_s^K(A, q) \cdot \{A/q\} \cdot \prod_{j=0}^{s-1} \{Aq^{r+j}\} \quad (3)$$

We introduced here an additional parameter h , distinguishing the "level" of differential expansion. HOMFLY polynomial itself arises at $h = 1$:

$$H_r^K(A, q^2) = \mathcal{H}_r^K(A, q^2|h = 1) \quad (4)$$

1 Relations at the roots of unity. Symmetric representations

In this paper we study *another* generalization of (2), which preserves its *factorized* form, but is instead true only at particular values of q – namely, at roots of unity. It turns out that for $q^{2m} = 1$

$$\boxed{H_{r+m}^{\mathcal{K}} = H_r^{\mathcal{K}} \cdot H_m^{\mathcal{K}} \Big|_{q^{2m}=1}} \quad (5)$$

where $H_r^{\mathcal{K}} = H_{[r]}^{\mathcal{K}}$ is HOMFLY polynomial in the totally symmetric representation $[r]$ (i.e. Young diagram is a single line of length r).

As an example take the knot $\mathcal{K} = 6_2$ from the Rolfsen table at $m = 2$. For $q = \pm 1$ we have (2) with $\sigma_{[1]}^{6_2} = H_1^{6_2} \Big|_{q=\pm 1} = \frac{1-2A^2+2A^4}{A^4}$. At the other two roots $q = \pm i$ we get:

$$\begin{aligned} H_1^{6_2} \Big|_{q=\pm i} &= -\frac{2A^4 + 6A^2 + 3}{A^4} \\ H_2^{6_2} \Big|_{q=\pm i} &= \frac{2A^8 - 2A^4 + 1}{A^8} \\ H_3^{6_2} \Big|_{q=\pm i} &= -\frac{(2A^4 + 6A^2 + 3)(2A^8 - 2A^4 + 1)}{A^{12}} \\ H_4^{6_2} \Big|_{q=\pm i} &= \frac{(2A^8 - 2A^4 + 1)^2}{A^{16}} \\ H_5^{6_2} \Big|_{q=\pm i} &= -\frac{(2A^4 + 6A^2 + 3)(2A^8 - 2A^4 + 1)^2}{A^{20}} \\ H_6^{6_2} \Big|_{q=\pm i} &= \frac{(2A^8 - 2A^4 + 1)^3}{A^{24}} \\ &\dots \end{aligned}$$

also in full accordance with (5).

Original (2) is now a particular case of (5) with $m = 1$. Since transposition of Young diagram $R \rightarrow \tilde{R}$ is equivalent to the substitution $q \rightarrow q^{-1}$ [17],

$$H_{\tilde{R}}^{\mathcal{K}}(q^2, A) = H_R^{\mathcal{K}}\left(\frac{1}{q^2}, A\right) \quad (6)$$

the same recursion holds for totally antisymmetric representations $[1^r]$ (Young diagram is a column of length r):

$$H_{[1^{r+m}]}^{\mathcal{K}} = H_{[1^r]}^{\mathcal{K}} \cdot H_{[1^m]}^{\mathcal{K}} \Big|_{q^{2m}=1} \quad (7)$$

In fact, (5) is equivalent to a more symmetric statement:

$$\boxed{\left(H_{r+m}^{\mathcal{K}} - H_r^{\mathcal{K}} \cdot H_m^{\mathcal{K}}\right) : \frac{\{q^r\}\{q^m\}}{\{q^{\gcd(r,m)}\}} \cdot \{A/q\}} \iff \boxed{\left(H_{[1^{r+m}]}^{\mathcal{K}} - H_{[1^r]}^{\mathcal{K}} \cdot H_{[1^m]}^{\mathcal{K}}\right) : \frac{\{q^r\}\{q^m\}}{\{q^{\gcd(r,m)}\}} \cdot \{Aq\}} \quad (8)$$

where $\gcd(r, m)$ is the greatest common divisor of r and m , and $\{x\} = x - x^{-1}$, so that the quantum number $[p] = \{q^p\}/\{q\}$ and for coprime r and m the r.h.s. is just $[r][m] \cdot \{q\}\{A/q\}$. The statement is that the r.h.s. factors out from the difference at the l.h.s. at any A and q .

It can be interpreted as one more property of the differential expansion (3):

$$\begin{aligned} \mathcal{H}_{r+m}^{\mathcal{K}} - \mathcal{H}_r^{\mathcal{K}} \cdot \mathcal{H}_m^{\mathcal{K}} = & \sum_{s=1}^{r+m} h^s \cdot \left(\frac{[r+m]!}{[s]![r+m-s]!} \prod_{j=0}^{s-1} \{Aq^{r+m+j}\} - \frac{[r]!}{[s]![r-s]!} \prod_{j=0}^{s-1} \{Aq^{r+j}\} - \frac{[m]!}{[s]![m-s]!} \prod_{j=0}^{s-1} \{Aq^{m+j}\} \right) \cdot \{A/q\} \cdot G_s^{\mathcal{K}} + \\ & + \sum_{s', s''=1} h^{s'+s''} \cdot \frac{[r]![m]!}{[s']![s'']![r-s']![m-s'']!} \prod_{j=0}^{s'-1} \{Aq^{r+j}\} \prod_{j=0}^{s''-1} \{Aq^{m+j}\} \cdot \{A/q\}^2 \cdot G_{s'}^{\mathcal{K}} G_{s''}^{\mathcal{K}} \quad (9) \end{aligned}$$

Many terms in these sums are immediately proportional to the r.h.s. of (8), but not all. Even the factor $\{q\}$ at $m = 1$ is not immediately obvious from (9), but at $q = \pm 1$ identity (2) can be additionally used:

$$(2) \implies G_s = \{A\}^{s-1} \cdot G_1^s \Big|_{q^2=1} \quad (10)$$

Still it turns out – and this is a highly non-trivial additional fact – that **proportionality to the r.h.s. of (8) holds independently** at each level $s = s' + s''$, i.e. **in each order of the h -expansion**, thus *enhancing* (8) to a whole set of quadratic restrictions on the values of $G_s^\mathcal{K}$ at roots of unity:

$$\begin{aligned}
& G_3^\mathcal{K} = q^2 \cdot \{A/q\} \cdot G_2^\mathcal{K} G_1^\mathcal{K} \Big|_{q^4=1} \\
& \dots \\
& \boxed{\prod_{j=0}^{m-1} \{Aq^{2r+j}\} \cdot G_{r+m}^\mathcal{K} = q^{rm} \cdot \{A/q\} \prod_{j=0}^{m-1} \{Aq^j\} \cdot G_r^\mathcal{K} G_m^\mathcal{K}} \Big|_{q^{2m}=1} \\
& \dots
\end{aligned} \tag{11}$$

The nicely-looking relation in the box arises at the order h^{m+r} , but it does not exhaust the set of relations: there are many more, arising at smaller powers of h in between $\max(r, m)$ and $r + m$, but they look less elegant.

A useful corollary of (11) is

$$G_{ms} + q^{m^2 s(s-1)/2} \cdot \{A/q\}^{s-1} \cdot (G_s)^m \vdash q^{2m} - 1 \tag{12}$$

2 Beyond symmetric representations

To really be a *generalization* of (2), relations like (5) should hold for arbitrary representations R , not only symmetric. Indeed it looks like there are plenty of them, and they continue to respect the grading by the level (number of boxes in Young diagram) – all such relations at special values of q are homogeneous in this grading. However it is difficult to find the reliable general rule. In this section we describe the relations at low levels $|R|$ and formulate a plausible general conjecture.

2.1 Extension to [21]

Beyond symmetric representations the story is more complicated, because the analogue of differential expansion (3) is still unknown. Moreover, not much is known about the non-symmetrically HOMFLY at all, even examples are restricted mostly to torus knots. The latest breakthrough in [5] provides answers for rather general knots, but only for $R = [21]$. Still, this very restricted result allows us to move further.

From the data, obtained on the lines of [5] we conclude empirically that the relevant generalization of (2) for $R = [21]$ is to arbitrary roots of order 6:

$$\boxed{H_{[21]}^\mathcal{K} = H_{[3]}^\mathcal{K} = H_{[111]}^\mathcal{K}} \Big|_{q^6=1} \iff \left(H_{[21]}^\mathcal{K} - H_{[3]}^\mathcal{K} \right) \vdash \{q^3\}\{A\} \tag{13}$$

Moreover, the second equality in (13) has its own generalization:

$$\boxed{\left(H_{[r]}^\mathcal{K} - H_{[1r]}^\mathcal{K} \right) \vdash [r][r-1]\{q\}\{A\}} \implies H_{[r]} = H_{[1r]} \Big|_{q^{2r}=1}, \quad H_{[r]} = H_{[1r]} \Big|_{q^{2r-2}=1} \tag{14}$$

Unfortunately (13) is all what we can check at this moment for rather general knots. In order to move further in non-symmetric case, we need to take a more risky road.

2.2 Implications from torus knots

After the very phenomenon is revealed from analysis of a rather general data, it can be further investigated on a far more restricted data field. Namely, if we believe/assume that there are *universal* relations between colored HOMFLY at roots of unity, i.e. valid for *all* knots, their concrete shape can be found by looking at particular knot families. Reliability of such statements is, of course, restricted, and what we get in this way are just *conjectures*. Still they can be brought to a relatively nice form and it is plausible that they are *universally* true.

Torus knots provide a natural family to look at, because this is the only one, where colored HOMFLY are available in arbitrary representation. This is because torus knots are more representation-theory than topological objects, and one should be very careful when extending observations made for this family to generic case – still we believe that conjectures below have good chances to be universally reliable.

HOMFLY for torus knot $[m, n]$ is given by the Rosso-Jones formula [11, 21, 17]

$$H_R^{[m, n]} = \sum_{|Q|=m|R|} c_R^Q q^{2n/m\kappa_Q} \chi_Q / \chi_R, \tag{15}$$

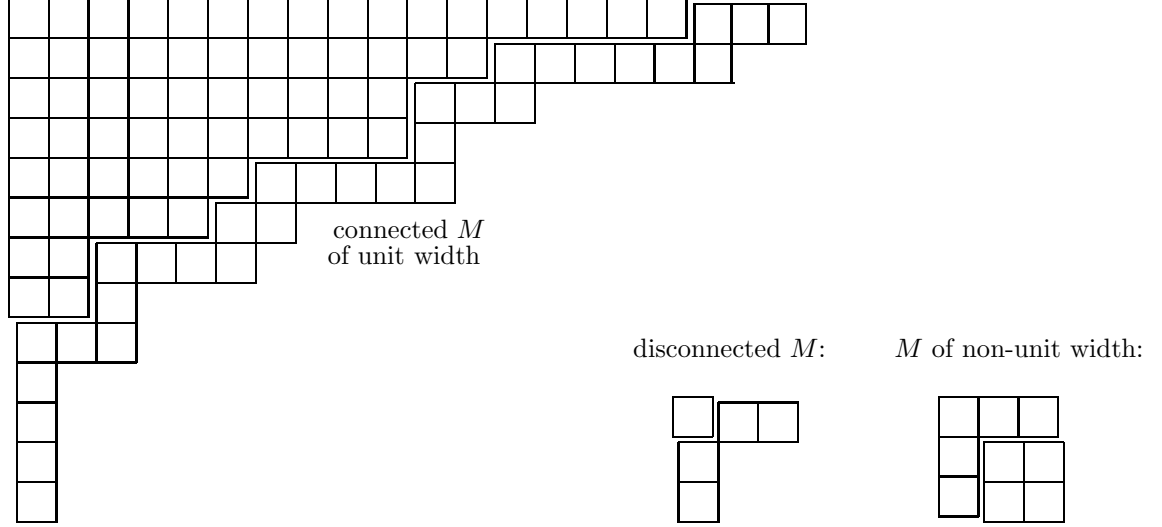
where κ_Q is the content of the diagram, and c_R^Q are matrix elements of the so-called Adams endomorphism in the basis of Schur polynomials.

2.3 Conjecture

From the study of torus knots we make a **conjecture**, which is presumably valid for all knots:

$$H_{R+M}^{\mathcal{K}} \stackrel{?}{=} H_m^{\mathcal{K}} \cdot H_R^{\mathcal{K}} \Big|_{q^{2m}=1} \quad \forall \text{ connected skew diagram } M \text{ of width one with } |M| = m \quad (16)$$

provided both R and $R+M$ are Young diagrams. The following picture is an explanation of what we mean by $R+M$:



Both conditions in (16) are necessary. The simple counterexamples are:

- $H_{[311]} \neq H_4 H_{[1]} \neq H_{[2]}^2 H_{[1]} \Big|_{q^8=1}$ – connectedness is indeed needed and
- $H_{[333]} \neq H_4 H_{[311]} \neq H_{[22]} H_{[311]} \Big|_{q^8=1}$ – width 2 is too much

Note that (14) implies that $H_{[m]} = H_{[1^m]}$ at $q^{2m} = 1$ – thus both can play the role of H_m in (16). Alternatively one can say that (14) is a particular case of the more general conjecture (16) – at the moment the difference is that the former is checked for all kinds of knots, while the latter – only for torus ones.

2.4 Examples of (14)+(16)

To illustrate the implications of (16) – or, if one prefers, the evidence in support of it – we now provide a few simple examples. We checked that all these relations are indeed true for a big variety of torus knots – and, as explained in above sec.2.2 we believe that they hold for all other knots, though this remains to be checked when the corresponding colored polynomials become available.

2.4.1 Factorization at $q^4 = 1$

In this case there are two options for M : $[2]$ and $[11]$ – a line and a column on length $m = 2$. Adding these two elements to various R we obtain from (16):

- Level $|R| + |M| = 3$: $\square \sim \square\square \sim \begin{smallmatrix} \square \\ \square \end{smallmatrix}$

$$H_3 = H_{[111]} = H_1 H_2 \neq H_{[21]} \quad (17)$$

Coincidence between H_3 and $H_{[111]}$ is also a corollary of the factor $[r-1]$ in (14).

- Level 4:

Here we have two parent diagrams $R = [2]$ and $R = [11]$, but their HOMFLY are identical at $q^4 = 1$, $H_{[2]} = H_{[11]}$ due to (14). Thus from (16)

$$H_Q = H_2^2 \quad \forall Q, \quad |Q| = 4 \quad (18)$$

- Level 5:

$$\begin{aligned} H_5 = H_{[32]} = H_{[311]} = H_{[221]} = H_{[1^5]} = H_1 H_2^2 \\ H_{[41]} = H_{[2111]} = H_{[21]} H_2 \end{aligned} \quad (19)$$

- Level 6:

$$H_Q = H_2^3 \quad \forall Q \neq [321] \quad |Q| = 6 \quad (20)$$

- Level 7:

$$\begin{aligned} H_7 = H_{[52]} = H_{[511]} = H_{[421]} = H_{[331]} = H_{[322]} = H_{[3211]} = H_{[311111]} = H_{[22111]} = H_{[1^7]} = H_1 H_2^2; \\ H_{[61]} = H_{[43]} = H_{[4111]} = H_{[2221]} = H_{[211111]} = H_{[21]} H_2^2 \end{aligned} \quad (21)$$

- Level 8:

$$\begin{aligned} H_8 = H_{[71]} = H_{[62]} = H_{[611]} = H_{[53]} = H_{[5111]} = H_{[44]} = H_{[431]} = H_{[422]} = H_{[4211]} = H_{[41111]} = \\ = H_{[332]} = H_{[3311]} = H_{[3221]} = H_{[311111]} = H_{[2222]} = H_{[22211]} = H_{[221111]} = H_{[2111111]} = H_{[1^8]} = H_2^4; \\ H_{[521]} = H_{[32111]} = H_{[321]} H_2 \end{aligned} \quad (22)$$

2.4.2 Factorization at $q^6 = 1$

First of all, we have universally valid relation (6), which is also applicable at roots of unity, where inversion gets equivalent to complex conjugation: $q^{-1} = q^*$. Thus,

- Level 2:

$$H_2 = H_{[11]}^* \quad (23)$$

Note, however, that $*$ is not supposed to act on A : $A^* = A$

Next, from (13):

- Level 3:

$$H_3 = H_{[21]} = H_{[111]} \quad (24)$$

Now we have for options four skew diagrams M of length $|M| = 3$: $[3]$, $[111]$, $[21]$ and $\overline{[21]}$ where the last one is an upside-down version $\overline{[21]} = \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ of $[21] = \begin{smallmatrix} \square & \square \end{smallmatrix}$. Adding them to various R we obtain the following "orbits":

- Level 4:

$$\begin{aligned} H_4 = H_{[22]} = H_{[1111]} = H_1 H_3 \\ H_{[31]} = H_{[211]}^* \end{aligned} \quad (25)$$

Coincidence between H_4 and $H_{[1111]}$ follows directly from (14), but it is also an implication of (16) – when we add either $M = [3]$ or $M = [111]$ to $R = [1]$. If $M = \overline{[21]}$ is added instead, we get $H_{[22]}$. However, there is no way to get $H_{[31]} \stackrel{(6)}{=} H_{[211]}^*$ and it is indeed independent.

- Level 5:

$$\begin{aligned} H_5 = H_{[221]} = H_{[2111]} = H_2 H_3 \\ H_{[41]} = H_{[3,2]} = H_{[11111]} = H_{[11]} H_3 \end{aligned} \quad (26)$$

- Level 6:

$$\begin{aligned} H_6 = H_{[51]} = H_{[411]} = H_{[33]} = H_{[321]} = H_{[3111]} = H_{[222]} = H_{[21111]} = H_{[111111]} = H_3^2 \\ H_{[42]} = H_{[2211]}^* \end{aligned} \quad (27)$$

- Level 7:

$$\begin{aligned}
H_7 = H_{[52]} = H_{[43]} = H_{[421]} = H_{[4111]} = H_{[3211]} = H_{[2221]} = H_{[22111]} = H_{[11111111]} = H_3^2 H_1 \\
H_{[61]} = H_{[322]} = H_{[31111]} = H_3 H_{[31]} \\
H_{[511]} = H_{[331]} = H_{[211111]} = H_{[211]} H_3
\end{aligned} \tag{28}$$

• Level 8:

$$\begin{aligned}
H_8 = H_{[53]} = H_{[521]} = H_{[5111]} = H_{[431]} = H_{[3311]} = H_{[2222]} = H_{[221111]} = H_{[2111111]} = H_3^2 H_2 \\
H_{[71]} = H_{[62]} = H_{[44]} = H_{[422]} = H_{[41111]} = H_{[3221]} = H_{[32111]} = H_{[22211]} = H_{[11111111]} = H_3^2 H_{[11]} \\
H_{[611]} = H_{[332]} = H_{[311111]} = H_3 H_{[311]} \\
H_{[4211]}
\end{aligned} \tag{29}$$

2.4.3 Factorizations at $q^8 = 1$

• Level 4:

$$\begin{aligned}
H_4 = H_{[31]} = H_{[211]} = H_{[1111]} \\
H_{[22]}
\end{aligned} \tag{30}$$

• Level 5:

$$\begin{aligned}
H_5 = H_{[32]} = H_{[221]} = H_{[11111]} = H_4 H_1 \\
H_{[41]} = H_{[2111]}^* \\
H_{[311]}
\end{aligned} \tag{31}$$

• Level 6:

$$\begin{aligned}
H_6 = H_{[33]} = H_{[2211]} = H_{[21111]} = H_4 H_2 \\
H_{[51]} = H_{[42]} = H_{[222]} = H_{[111111]} = H_4 H_{[11]} \\
H_{[411]} = H_{[3111]}^* \\
H_{[321]}
\end{aligned} \tag{32}$$

• Level 7:

$$\begin{aligned}
H_7 = H_{[331]} = H_{[3211]} = H_{[31111]} = H_4 H_3 \\
H_{[61]} = H_{[43]} = H_{[2221]} = H_{[211111]} = H_{[21]} H_4 \\
H_{[52]} = H_{[22111]}^* \\
H_{[511]} = H_{[421]} = H_{[322]} = H_{[1111111]} = H_4 H_{[111]} \\
H_{[4111]}
\end{aligned} \tag{33}$$

• Level 8:

$$\begin{aligned}
H_8 = H_{[71]} = H_{[611]} = H_{[5111]} = H_{[44]} = H_{[431]} = H_{[4211]} = H_{[41111]} = H_{[332]} = H_{[3221]} = \\
= H_{[311111]} = H_{[2222]} = H_{[2111111]} = H_{[11111111]} = H_4^2 \\
H_{[62]} = H_{[53]} = H_{[22211]} = H_{[221111]} = H_4 H_{[22]} \\
H_{[422]} = H_{[3311]}^* \\
H_{[521]} = H_{[32111]}^*
\end{aligned} \tag{34}$$

2.4.4 Towards colored $\hat{\mathcal{A}}$ polynomials

Another way to represent these relations is to consider factorization of the differences between HOMFLY in different representations, for example

$$H_{[41]} - H_{[21]} H_{[11]} \quad \vdots \quad \frac{\{q^3\}\{q^2\}}{\{q\}} \quad \Leftarrow \quad (16) + (14) + (13) \tag{35}$$

describes simultaneously several relations from above list. Reformulating (16) in such form can be the first step towards derivation of \mathcal{A} -polynomial-like equations [25, 22, 26, 27, 36] for the colored knots.

In such differences there can be also factors containing $\{Aq^j\}$, which imply additional relations at special values of N – they are also of interest. We provide some examples in the Appendix at the end of this paper.

3 Reduction of Kashaev polynomial to the special one

Discussing knot polynomials at roots of unity it is difficult to avoid looking at Kashaev polynomial [38]

$$K_R^\mathcal{K}(A) = H_R^\mathcal{K} \left(q^2 = e^{2\pi i/|R|}, A \right) \quad (36)$$

which is the value of colored HOMFLY at a primitive root of unity $q^{2|R|} = 1$. Doing so, we observe a remarkable fact: for all single hook diagrams R Kashaev polynomial is easily expressed through the special polynomial:

$$\boxed{K_R^\mathcal{K}(A) = K_{[1]}^\mathcal{K} \left(A^{|R|} \right) = H_{[1]}^\mathcal{K} \left(q^2 = 1, A^{|R|} \right) = \sigma_{[1]}^\mathcal{K} \left(A^{|R|} \right)} \quad \forall R = [r, 1^k] \quad (37)$$

It looks like an $A - q$ dual of the mysterious relation [22, 23] for the Alexander polynomial $Al_R^\mathcal{K}(q) = H_R(A = 1, q)$:

$$Al_R^\mathcal{K}(q) = Al_{[1]}^\mathcal{K} \left(q^{|R|} \right) \quad \forall R = [r, 1^k] \quad (38)$$

This means that Kashaev polynomial for single-hook diagrams – and thus for all symmetric representations, where it is mostly used – is actually nothing more than the special one. But, like Alexander, it becomes highly non-trivial whenever the number of hooks exceeds one.

For example, for torus knot $\mathcal{K} = [3, 7]$

$$\begin{aligned} K_{[1]}^{[3,7]} &= \frac{5 - 16A^2 + 12A^4}{A^{16}} \\ K_{[2]}^{[3,7]} &= \frac{5 - 16A^4 + 12A^8}{A^{32}} = K_{[1]}^{[3,7]}(A^2) \\ K_{[1,1]}^{[3,7]} &= \frac{5 - 16A^4 + 12A^8}{A^{32}} = K_{[1]}^{[3,7]}(A^2) \\ K_{[3]}^{[3,7]} &= \frac{5 - 16A^6 + 12A^{12}}{A^{48}} = K_{[1]}^{[3,7]}(A^3) \\ K_{[2,1]}^{[3,7]} &= \frac{5 - 16A^6 + 12A^{12}}{A^{48}} = K_{[1]}^{[3,7]}(A^3) \\ K_{[1,1,1]}^{[3,7]} &= \frac{5 - 16A^6 + 12A^{12}}{A^{48}} = K_{[1]}^{[3,7]}(A^3) \\ K_{[4]}^{[3,7]} &= \frac{5 - 16A^8 + 12A^{16}}{A^{64}} = K_{[1]}^{[3,7]}(A^4) \\ K_{[3,1]}^{[3,7]} &= \frac{5 - 16A^8 + 12A^{16}}{A^{64}} = K_{[1]}^{[3,7]}(A^4) \\ K_{[2,2]}^{[3,7]} &= \frac{17 + 16A^2 + 20A^4 + 24A^6 - 40A^8 - 56A^{10} - 68A^{12} - 80A^{14} - 24A^{16}}{A^{64}} \\ K_{[2,1,1]}^{[3,7]} &= \frac{5 - 16A^8 + 12A^{16}}{A^{64}} = K_{[1]}^{[3,7]}(A^4) \\ K_{[1,1,1,1]}^{[3,7]} &= \frac{5 - 16A^8 + 12A^{16}}{A^{64}} = K_{[1]}^{[3,7]}(A^4) \end{aligned}$$

For symmetric representations (37) can be reformulated as the property of the differential expansion. Eq.(3) implies that Kashaev polynomial is a sum of just two terms:

$$K_r^\mathcal{K}(A) = 1 + G_r^\mathcal{K}(A, q) \cdot \{A/q\} \cdot \prod_{j=0}^{r-1} \{Aq^{r+j}\} \Bigg|_{q=e^{i\pi/r}} \quad (39)$$

All other terms vanish, because $[r] = 0$ at $q = e^{i\pi/r}$, what nullifies binomial coefficients.

The product

$$\prod_{j=0}^{r-1} \{Aq^{r+j}\} \Bigg|_{q=e^{i\pi/r}} = e^{-i\pi(r+1)/2} \{A^r\} \quad (40)$$

If \mathcal{K} has *defect* zero [37], i.e. $G_r^\mathcal{K} \cdot \{A/q\} = F_r^\mathcal{K} \cdot \prod_{j=0}^{r-1} \{Aq^{j-1}\}$, then we get the second product of the same kind and

$$K_r^\mathcal{K}(A) = 1 + F_r \left(q = e^{i\pi/r}, A \right) \cdot \{A^r\}^2 \quad \text{provided } \delta^\mathcal{K} = 0 \quad (41)$$

so that (37) means

$$F_r \left(q = e^{i\pi/r}, A \right) = F_1 \left(q^2 = 1, A^r \right) \quad (42)$$

what is indeed true, for example, for 4_1 with all $F_r = 1$, for 3_1 with $F_r = (-)^r q^{-r(r-1)} A^{-2r}$ and – a little less trivially – for other twist knots [28], which all have defect zero.

For generic knot with arbitrary defect $\delta^\mathcal{K}$ eq.(37) implies that

$$e^{-i\pi(r+1)/2} \cdot G_r \left(q = e^{i\pi/r}, A \right) \cdot \{Ae^{-i\pi/r}\} = G_1(q = 1, A^r) \cdot \{A^r\}. \quad (43)$$

Non-primitive roots of degree $2m$ are primitive of some lower degree, thus the form of (37) and (43) is not universal for all the roots $q^{2m} = 1$ – in variance with (11).

Note in passing that since and $Al_{[1]}^\mathcal{K}(q = \pm 1) = 1$, eq.(38) implies that

$$Al_{[r, 1^k]}^\mathcal{K}(q) = 1 \Big|_{q^{2(r+k)}=1} \quad (44)$$

i.e. Alexander polynomial is just trivial at the relevant root of unity – and for primitive root this can be also considered as an implication of (37) at $A = 1$.

4 Conclusion

In this paper we reported a number of interesting relations for colored HOMFLY polynomial at the roots of unity, which seem true for arbitrary knots. They are obtained experimentally, by looking at explicit expressions, implied by the recent [4, 5] for a vast variety of knots. Proofs are not yet available, and evidence can be not fully convincing, especially for non-symmetric representations, where it comes from the torus knots only. Remarkable conjectures (16) and (37) cry for applying new efforts to the study of colored polynomials – what in the frame of [5] basically requires an effort in calculating Racah matrices. A possible way to conceptual proofs can be within the Cherednik's DAHA approach [39, 18, 29], where something special also happens when q is a root of unity [40]. Even more distinguished are the roots of unity in the original method of [16]. As explained in sec.2.4.4 conjecture (16), if adequately reformulated, is already sufficient to study the colored $\hat{\mathcal{A}}$ -polynomials – and this is another exciting direction, opened by our results.

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Appendix. Possible germs of colored $\hat{\mathcal{A}}$ -polynomials

In this appendix we open a somewhat different line of factorization properties, where what factors out from the differences between colored polynomials are factors $\{A/q^k\}$ with some k . When such factor appears, it means that the two HOMFLY polynomials coincide when $A = q^k$. Relations of this type can be equally important as those at roots of unity – at least for the search of colored $\hat{\mathcal{A}}$ -polynomials.

What we give is just a beginning of such list, and we do not formulate any conjecture, comparable in generality to (16) – this is left for the future.

$$\begin{aligned}
H_{[21]}^{\mathcal{K}} - H_{[1]}^{\mathcal{K}} & \quad \vdots \quad \{A/q^2\}\{Aq^2\} \\
H_{[21]}^{\mathcal{K}} - H_{[3]}^{\mathcal{K}} & \quad \vdots \quad \{A\}\{q^3\} \\
H_{[21]}^{\mathcal{K}} - H_{[111]}^{\mathcal{K}} & \quad \vdots \quad \{A\}\{q^3\} \\
\\
H_{[21]}^{\mathcal{K}} - H_{[32]}^{\mathcal{K}} & \quad \vdots \quad \{A/q^2\}\{Aq^4\} \\
H_{[21]}^{\mathcal{K}} - H_{[221]}^{\mathcal{K}} & \quad \vdots \quad \{Aq^2\}\{A/q^4\} \\
H_{[21]}^{\mathcal{K}} - H_{[321]}^{\mathcal{K}} & \quad \vdots \quad \{A/q^3\}\{Aq^3\}
\end{aligned} \tag{45}$$

The next table is made closer in style to sec.2.4: instead of writing a factor $\{Aq^{-k}\}$ we list coincidences at $A = q^k$.

At $A = 1$ relations are only between representations of the same size – and these were listed in sec.2.4.

At $A = q$ all symmetric representations are trivial $H_r^{\mathcal{K}}(q = 1, A) = 1$ due to the factors $\{A/q\}$ in (3), additional relations between our *reduced* (and thus non-vanishing) colored HOMFLY are listed in the first column:

